Variational Methods in Image Processing

ÚTIA AV ČR
1 Introduction
- Motivation
- Derivation of Euler-Lagrange Equation
- Variational Problem and P.D.E.
Outline

1. Introduction
   • Motivation
   • Derivation of Euler-Lagrange Equation
   • Variational Problem and P.D.E.
The Brachistochrone Problem:

“Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.”

Johann Bernoulli in 1696
The Brachistochrone Problem:

“Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.”

Johann Bernoulli in 1696
The Brachistochrone Problem:

“Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.”

Johann Bernoulli in 1696

In one year Newton, Johann and Jacob Bernoulli, Leibniz, and de L'Hôpital came with the solution.
The problem was generalized and an analytic method was given by Euler (1744) and Lagrange (1760).
Most of the image processing tasks can be formulated as optimization problems, i.e., minimization of functionals.
Most of the image processing tasks can be formulated as optimization problems, i.e., minimization of functionals.

Calculus of Variations solves

$$\min_u F(u(x)),$$

where $u \in X$,
$F : X \rightarrow \mathbb{R}$,
$X \ldots$ Banach space
Most of the image processing tasks can be formulated as optimization problems, i.e., minimization of functionals. Calculus of Variations solves

$$\min_u F(u(x)),$$

where $u \in X$, $F : X \rightarrow R$, $X \ldots$ Banach space.

Solution by means of Euler-Lagrange (E-L) equation.
Calculus of Variations

Integral functionals

\[ F(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx \]

Example

- \( x \in \mathbb{R}^2 \) \ldots space of coordinates \([x_1, x_2]\)
- \( \Omega \ldots \) image support
- \( u(x) : \mathbb{R}^2 \rightarrow \mathbb{R} \) \ldots grayscale image
- \( \nabla u(x) \ldots \) image gradient \([u_{x_1}, u_{x_2}]\)
Examples

- Image Registration
  given a set of CP pairs \([x_i, y_i] \leftrightarrow [\tilde{x}_i, \tilde{y}_i]\)
  find \(\tilde{x} = f(x, y), \tilde{y} = g(x, y)\)

\[
F(f) = \sum_i (\tilde{x}_i - f(x_i, y_i))^2 + \lambda \int \int f_{xx}^2 + 2f_{xy}^2 + f_{yy}^2 \, dx \, dy
\]

and a similar equation for \(g(x, y)\)
Examples

- Image Registration
Examples

- **Image Registration**

- **Image Reconstruction**
  
  given an image acquisition model $H(\cdot)$ and measurement $g$
  find the original image $u$

  $$F(u) = \int (H(u) - g)^2 \, dx + \lambda \int |\nabla u|^2$$
Examples

- Image Registration
- Image Reconstruction
Examples

- **Image Segmentation**
  find a piece-wise constant representation $u$ of an image $g$

\[
F(u, K) = \int_{\Omega-K} (u - g)^2 \, dx + \alpha \int_{\Omega-K} |\nabla u|^2 \, dx + \beta \int_K ds
\]
Examples

- Image Segmentation
Examples

- **Image Segmentation**

- **Motion Estimation**
  
  find velocity field \( v(x) \equiv [v_1(x), v_2(x)] \) in an image sequence \( u(x, t) \)

\[
F(v) = \int |v \cdot \nabla u + u_t| \, dx + \alpha \sum_j \int |\nabla v_j| \, dx + \beta \int c(\nabla u)|v|^2 \, dx
\]
Examples

- **Image Segmentation**
- **Motion Estimation**
Examples

- Image classification
Examples

- Image classification
- and many more
Outline

1. Introduction
   - Motivation
   - Derivation of Euler-Lagrange Equation
   - Variational Problem and P.D.E.
Extrema points

From the differential calculus follows that
From the differential calculus follows that if $x$ is an extremum of $g(x) : \mathbb{R}^N \rightarrow \mathbb{R}$ then $\forall \nu \in \mathbb{R}^N$
From the differential calculus follows that if $x$ is an extremum of $g(x) : R^N \rightarrow R$ then $\forall \nu \in R^N$

$$\frac{d}{d\varepsilon} g(x + \varepsilon \nu) \bigg|_{\varepsilon=0} = 0$$
Extrema points

From the differential calculus follows that if $x$ is an extremum of $g(x) : \mathbb{R}^N \rightarrow \mathbb{R}$ then $\forall \nu \in \mathbb{R}^N$

$$
\frac{d}{d\varepsilon} g(x + \varepsilon \nu) \bigg|_{\varepsilon=0} = 0 \quad = \quad \langle \nabla g(x), \nu \rangle
$$
Extrema points

From the differential calculus follows that if $x$ is an extremum of $g(x) : \mathbb{R}^N \to \mathbb{R}$ then $\forall \nu \in \mathbb{R}^N$

$$\frac{d}{d\varepsilon} g(x + \varepsilon \nu) \bigg|_{\varepsilon=0} = 0 \quad = \quad \langle \nabla g(x), \nu \rangle \quad \Leftrightarrow \quad \nabla g(x) = 0$$
Extrema points

From the differential calculus follows that if $x$ is an extremum of $g(x) : \mathbb{R}^N \rightarrow \mathbb{R}$ then $\forall \nu \in \mathbb{R}^N$

$$\frac{d}{d\varepsilon} g(x + \varepsilon\nu) \bigg|_{\varepsilon=0} = 0 = \langle \nabla g(x), \nu \rangle \iff \nabla g(x) = 0$$

in 1-D ($g : \mathbb{R} \rightarrow \mathbb{R}$) we get the classical condition

$$g'(x) = 0$$
Variation of Functional

\[ F(u) = \int_{a}^{b} f(x, u, u') \, dx \]

If \( u \) is extremum of \( F \) then from differential calculus follows

\[
d \left. \frac{\partial F}{\partial \varepsilon} \right|_{\varepsilon=0} = 0 \quad \forall \ v
\]

\[ F(u + \varepsilon v) = \int_{a}^{b} f(x, u + \varepsilon v, (u + \varepsilon v)') \, dx \]
Variation of Functional

\[ F(u) = \int_{a}^{b} f(x, u, u') \, dx \]

if \( u \) is extremum of \( F \) then from differential calculus follows

\[ \frac{d}{d\varepsilon} F(u + \varepsilon v) \bigg|_{\varepsilon=0} = 0 \quad \forall v \]

\[ F(u + \varepsilon v) = \int_{a}^{b} f(x, u + \varepsilon v, u' + \varepsilon v') \, dx \]
Chain Rule

\[
\frac{d}{dx} f(u(x), v(x)) = \left( \frac{\partial}{\partial u} f(u, v) \right) \frac{du}{dx} + \left( \frac{\partial}{\partial v} f(u, v) \right) \frac{dv}{dx}
\]
Chain Rule

\[ \frac{d}{dx} f(u(x), v(x)) = \left( \frac{\partial}{\partial u} f(u, v) \right) \frac{du}{dx} + \left( \frac{\partial}{\partial v} f(u, v) \right) \frac{dv}{dx} \]

**Example**

\[ u = x, \quad v = \sin x, \quad f = uv = x \sin x \]

\[ \frac{d}{dx} f(u, v) = v(x)1 + u(x) \cos x = \sin x + x \cos x \]
Chain Rule

\[ \frac{d}{dx} f(u(x), v(x)) = \left( \frac{\partial}{\partial u} f(u, v) \right) \frac{du}{dx} + \left( \frac{\partial}{\partial v} f(u, v) \right) \frac{dv}{dx} \]

Example

\[ u = x, \quad v = \sin x, \quad f = uv = x \sin x \]

\[ \frac{d}{dx} f(u, v) = v(x)1 + u(x) \cos x = \sin x + x \cos x \]
Chain Rule

\[
\frac{d}{dx} f(u(x), v(x)) = \left( \frac{\partial}{\partial u} f(u, v) \right) \frac{du}{dx} + \left( \frac{\partial}{\partial v} f(u, v) \right) \frac{dv}{dx}
\]

Example

\[u = x, \ v = \sin x, \ f = uv = x \sin x\]

\[
\frac{d}{dx} f(u, v) = v(x)1 + u(x) \cos x = \sin x + x \cos x
\]
Partial derivatives

Example

\[ f(x, u) = x \sin x \]

\[
\frac{df}{dx} = \text{chain rule} = \sin x + x \cos x
\]
Partial derivatives

Example

\[ f(x, u) = x \sin x \]

\[ \frac{df}{dx} = \text{chain rule} = \sin x + x \cos x \]

but

\[ \frac{\partial f}{\partial x} = \sin x \]
per partes

\[ \int_{a}^{b} uv' = uv \bigg|_{a}^{b} - \int_{a}^{b} u'v \]
Derivation of E-L equation

\[
\frac{d}{d\varepsilon} F(u + \varepsilon v) = \frac{d}{d\varepsilon} \int_{a}^{b} f(x, u + \varepsilon v, u' + \varepsilon v')
\]
Derivation of E-L equation

\[
\frac{d}{d\varepsilon} F(\varepsilon) = \frac{d}{d\varepsilon} \int_{a}^{b} f(x, \varepsilon, u' + \varepsilon v') \\
= \int_{a}^{b} \frac{\partial f}{\partial u} v' + \frac{\partial f}{\partial u'} v' 
\]

chain rule
Derivation of E-L equation

\[
\frac{d}{d\varepsilon} F(u + \varepsilon v) = \frac{d}{d\varepsilon} \int_{a}^{b} f(x, u + \varepsilon v, u' + \varepsilon v')
\]

\[
= \int_{a}^{b} \left( \frac{\partial f}{\partial u} v + \frac{\partial f}{\partial u'} v' \right)
\]

chain rule

\[
= \int_{a}^{b} \frac{\partial f}{\partial u} v - \int_{a}^{b} \frac{d}{dx} \frac{\partial f}{\partial u'} v + \frac{\partial f}{\partial u'} v \bigg|_{a}^{b}
\]

per partes
\[
\frac{d}{d\varepsilon} F(u + \varepsilon v) = \frac{d}{d\varepsilon} \int_a^b f(x, u + \varepsilon v, u' + \varepsilon v')
\]

\[= \int_a^b \frac{\partial f}{\partial u} v + \frac{\partial f}{\partial u'} v' \quad \text{chain rule}
\]
\[= \int_a^b \frac{\partial f}{\partial u} v - \int_a^b \frac{d}{dx} \frac{\partial f}{\partial u'} v + \frac{\partial f}{\partial u'} v \bigg|_a^b \quad \text{per partes}
\]
\[= \int_a^b \left[ \frac{\partial f}{\partial u} - \frac{d}{dx} \frac{\partial f}{\partial u'} \right] v + \frac{\partial f}{\partial u'} v \bigg|_a^b = 0
\]
Derivation of E-L equation

\[
\frac{d}{d\varepsilon} F(u + \varepsilon v) = \frac{d}{d\varepsilon} \int_{a}^{b} f(x, u + \varepsilon v, u' + \varepsilon v')
\]

\[
= \int_{a}^{b} \frac{\partial f}{\partial u} v + \frac{\partial f}{\partial u'} v' \quad \text{chain rule}
\]

\[
= \int_{a}^{b} \frac{\partial f}{\partial u} v - \int_{a}^{b} \frac{d}{dx} \frac{\partial f}{\partial u'} v + \frac{\partial f}{\partial u'} v \bigg|_{a}^{b} \quad \text{per partes}
\]

\[
= \int_{a}^{b} \left[ \frac{\partial f}{\partial u} - \frac{d}{dx} \frac{\partial f}{\partial u'} \right] v + \frac{\partial f}{\partial u'} v \bigg|_{a}^{b} = 0
\]

to be equal to 0 for any \( v \), \( \left[ \frac{\partial f}{\partial u} - \frac{d}{dx} \frac{\partial f}{\partial u'} \right] = 0 \rightarrow \text{E-L equation} \)
Derivation of E-L equation

\[ \frac{d}{d\varepsilon} F(u + \varepsilon v) = \frac{d}{d\varepsilon} \int_a^b f(x, u + \varepsilon v, u' + \varepsilon v') \]

\[ = \int_a^b \frac{\partial f}{\partial u} v + \frac{\partial f}{\partial u'} v' \quad \text{chain rule} \]

\[ = \int_a^b \frac{\partial f}{\partial u} v - \int_a^b \frac{d}{dx} \frac{\partial f}{\partial u'} v + \frac{\partial f}{\partial u'} v \bigg|_a^b \quad \text{per partes} \]

\[ = \int_a^b \left[ \frac{\partial f}{\partial u} - \frac{d}{dx} \frac{\partial f}{\partial u'} \right] v + \frac{\partial f}{\partial u'} v \bigg|_a^b = 0 \]

to be equal to 0, we need boundary conditions,
e.g., fixed \( u(a), u(b) \rightarrow v(a) = v(b) = 0 \).
Find the shortest path between points $A$ and $B$, assuming that one can write $y = y(x)$. 

$\int_{a}^{b} \sqrt{1 + (y'(x))^2} \, dx$ 

E-L eq.:

$- \frac{d}{dx} y'(x) \sqrt{1 + (y'(x))^2} = 0 \Rightarrow y'(x) = \text{constant}$

$y(x)$ is a straight line between $A$ and $B$. 

Variational Methods
Find the shortest path between points $A$ and $B$, assuming that one can write $y = y(x)$.

We want to minimize $F(y(x)) = \int_a^b \sqrt{1 + y'(x)^2} \, dx$ with b.c. $y(a) = \alpha$, $y(b) = \beta$. 
Find the shortest path between points A and B, assuming that one can write \( y = y(x) \).

We want to minimize \( F(y(x)) = \int_a^b \sqrt{1 + y'(x)^2} \, dx \) with b.c. \( y(a) = \alpha, \ y(b) = \beta \).

E-L eq.: \(- \frac{d}{dx} \frac{y'(x)}{\sqrt{1+y'(x)^2}} = 0\)
Find the shortest path between points $A$ and $B$, assuming that one can write $y = y(x)$.

We want to minimize $F(y(x)) = \int_a^b \sqrt{1 + y'(x)^2} \, dx$ with b.c. $y(a) = \alpha$, $y(b) = \beta$.

E-L eq.: $-\frac{d}{dx} \frac{y'(x)}{\sqrt{1 + y'(x)^2}} = 0 \implies y' = C \sqrt{1 + y'^2}$
**Toy case**

**Shortest path**

- Find the shortest path between points $A$ and $B$, assuming that one can write $y = y(x)$.

- We want to minimize $F(y(x)) = \int_{a}^{b} \sqrt{1 + y'(x)^2} \, dx$ with b.c. $y(a) = \alpha$, $y(b) = \beta$.

- E-L eq.: $-\frac{d}{dx} \frac{y'(x)}{\sqrt{1+y'(x)^2}} = 0 \Rightarrow y' = C \sqrt{1 + y'^2} \Rightarrow y' = \text{constant}$
Find the shortest path between points A and B, assuming that one can write \( y = y(x) \).

We want to minimize \( F(y(x)) = \int_a^b \sqrt{1 + y'(x)^2} \, dx \) with b.c. \( y(a) = \alpha \), \( y(b) = \beta \).

E-L eq.: \[-\frac{d}{dx} \frac{y'(x)}{\sqrt{1+y'(x)^2}} = 0 \Rightarrow y' = C \sqrt{1 + y'^2} \Rightarrow y' = \text{constant} \]

\( y(x) \) is a straight line between A and B.
If $u(x) : \mathbb{R}^N \rightarrow \mathbb{R}$ is extremum of $F(u) = \int_{\Omega} f(x, u, \nabla u) \, dx$, where $\nabla u \equiv [u_{x_1}, \ldots, u_{x_N}]$ then
If $u(x): \mathbb{R}^N \rightarrow \mathbb{R}$ is extremum of $F(u) = \int_{\Omega} f(x, u, \nabla u) \, dx$, where $\nabla u \equiv [u_{x_1}, \ldots, u_{x_N}]$ then

$$F'(u) = \frac{\partial f}{\partial u}(x, u, \nabla u) - \sum_{i=1}^{N} \frac{d}{dx_i} \left( \frac{\partial f}{\partial u_{x_i}}(x, u, \nabla u) \right) = 0,$$

which is the E-L equation.
Beltrami Identity

\( f(x, u, u') \)

\[ \frac{\partial f}{\partial u} - \frac{d}{dx} \left( \frac{\partial f}{\partial u'} \right) = 0 \]
Beltrami Identity

\[ f(x, u, u') \]
\[
\frac{df}{dx} = \frac{\partial f}{\partial u} u' + \frac{\partial f}{\partial u'} u'' + \frac{\partial f}{\partial x}
\]

\[
\frac{\partial f}{\partial u} - \frac{d}{dx} \left( \frac{\partial f}{\partial u'} \right) = 0
\]
Beltrami Identity

\[ f(x, u, u') \]

\[ \frac{df}{dx} = \frac{\partial f}{\partial u} u' + \frac{\partial f}{\partial u'} u'' + \frac{\partial f}{\partial x} \]

\[ \frac{\partial f}{\partial u} u' = \frac{df}{dx} - \frac{\partial f}{\partial u'} u'' - \frac{\partial f}{\partial x} \]

\[ \frac{\partial f}{\partial u} - \frac{d}{dx} \left( \frac{\partial f}{\partial u'} \right) = 0 \]

\[ u' \frac{\partial f}{\partial u} - u' \frac{d}{dx} \left( \frac{\partial f}{\partial u'} \right) = 0 \]
Beltrami Identity

\[ f(x, u, u') \]

\[
\frac{df}{dx} = \frac{\partial f}{\partial u} u' + \frac{\partial f}{\partial u'} u'' + \frac{\partial f}{\partial x}
\]

\[
\frac{\partial f}{\partial u} u' = \frac{df}{dx} - \frac{\partial f}{\partial u'} u'' - \frac{\partial f}{\partial x}
\]

\[
\frac{\partial f}{\partial u} u' - u' \frac{d}{dx} \left( \frac{\partial f}{\partial u'} \right) = 0
\]

Variational Methods
Beltrami Identity

\[ f(x, u, u') \]

\[ \frac{df}{dx} = \frac{\partial f}{\partial u} u' + \frac{\partial f}{\partial u'} u'' + \frac{\partial f}{\partial x} \]

\[ \frac{\partial f}{\partial u} u' = \frac{df}{dx} - \frac{\partial f}{\partial u'} u'' - \frac{\partial f}{\partial x} \]

\[ \frac{\partial f}{\partial u} - \frac{d}{dx} \left( \frac{\partial f}{\partial u'} \right) = 0 \]

\[ u' \frac{\partial f}{\partial u} - u' \frac{d}{dx} \left( \frac{\partial f}{\partial u'} \right) = 0 \]

\[ \frac{df}{dx} - \frac{\partial f}{\partial u'} u'' - \frac{\partial f}{\partial x} - u' \frac{d}{dx} \left( \frac{\partial f}{\partial u'} \right) = 0 \]

\[ \frac{d}{dx} \left( f - u' \frac{\partial f}{\partial u'} \right) - \frac{\partial f}{\partial x} = 0 \]
Beltrami Identity

\[ \frac{d}{dx} \left( f - u' \frac{\partial f}{\partial u'} \right) - \frac{\partial f}{\partial x} = 0 \]
Beltrami Identity

\[
\frac{d}{dx} \left( f - u' \frac{\partial f}{\partial u'} \right) - \frac{\partial f}{\partial x} = 0
\]

if \( \frac{\partial f}{\partial x} = 0 \) then

\[
\frac{d}{dx} \left( f - u' \frac{\partial f}{\partial u'} \right) = 0 \iff f - u' \frac{\partial f}{\partial u'} = C
\]
Brachistochrone

- \( F = \int dt \), \( \min F \) ... curve of the shortest time.
- \( F = \int \frac{ds}{v} = \int_0^b \frac{\sqrt{1+(y'(x))^2}}{v} dx \)
- \( \frac{1}{2}mv^2 = mgy(x) \Rightarrow v = \sqrt{2gy(x)} \)
Brachistochrone

- $F = \int dt$, $\min F$ ... curve of the shortest time.
- $F = \int \frac{ds}{v} = \int_0^b \frac{\sqrt{1+(y'(x))^2}}{v} \, dx$
- $\frac{1}{2}mv^2 = mgy(x) \Rightarrow v = \sqrt{2gy(x)}$
- $F = \int_0^b \frac{\sqrt{1+(y'(x))^2}}{\sqrt{2gy}} \, dx$
Brachistochrone

\[ f(y, y') = \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}} \]
Brachistochrone

\[ f(y, y') = \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}} \]

\[ f - y' \frac{\partial f}{\partial y'} = C \quad \text{Beltrami identity} \]
Brachistochrone

\[ f(y, y') = \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}} \]

\[ f - y' \frac{\partial f}{\partial y'} = C \quad \text{Beltrami identity} \]
Brachistochrone

\[ f(y, y') = \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}} \]

\[ f - y' \frac{\partial f}{\partial y'} = C \quad \text{Beltrami identity} \]

\[ y(1 + (y')^2) = \frac{1}{2gC^2} = k \]
Brachistochrone

$$f(y, y') = \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}}$$

$$f - y' \frac{\partial f}{\partial y'} = C \quad \text{Beltrami identity}$$

$$y(1 + (y')^2) = \frac{1}{2gC^2} = k$$

The solution is a cycloid

$$x(\theta) = \frac{1}{2}k(\theta - \sin \theta), \quad y(\theta) = \frac{1}{2}k(1 - \cos \theta)$$
Boundary conditions

- using “per partes” on $u(x, y)$, $\mathbf{n}(x, y) \equiv [n_1(x, y), n_2(x, y)]$
- normal vector at the boundary $\partial \Omega$

$$\frac{\partial}{\partial \varepsilon} F(u + \varepsilon v) = \int \cdot dx dy + \int_{\partial \Omega} \left[ \frac{\partial f}{\partial u_x} n_1 + \frac{\partial f}{\partial u_y} n_2 \right] v \, ds$$
Boundary conditions

- using “per partes” on \( u(x, y) \), \( \mathbf{n}(x, y) \equiv [n_1(x, y), n_2(x, y)] \)

normal vector at the boundary \( \partial \Omega \)

\[
\frac{\partial}{\partial \varepsilon} F(u + \varepsilon v) = \int (\cdot) dx dy + \int_{\partial \Omega} \left[ \frac{\partial f}{\partial u_x} n_1 + \frac{\partial f}{\partial u_y} n_2 \right] v \, ds
\]

- Dirichlet b.c.

\( u \) is predefined at the boundary \( \partial \Omega \rightarrow v(\partial \Omega) = 0 \)
Boundary conditions

- using “per partes” on $u(x, y)$, $n(x, y) \equiv [n_1(x, y), n_2(x, y)]$

  normal vector at the boundary $\partial \Omega$

  \[
  \frac{\partial}{\partial \varepsilon} F(u + \varepsilon v) = \int (\cdot) dx dy + \int_{\partial \Omega} \left[ \frac{\partial f}{\partial u_x} n_1 + \frac{\partial f}{\partial u_y} n_2 \right] v \, ds
  \]

- Dirichlet b.c.
  $u$ is predefined at the boundary $\partial \Omega \rightarrow v(\partial \Omega) = 0$

- Neumann b.c.
  derivative in the direction of normal $\frac{\partial u}{\partial n} = 0$
Boundary conditions

- using “per partes” on $u(x, y)$, $n(x, y) \equiv [n_1(x, y), n_2(x, y)]$
- normal vector at the boundary $\partial \Omega$

$$\frac{\partial}{\partial \varepsilon} F(u + \varepsilon v) = \int (\cdot) dxdy + \int_{\partial \Omega} \left[ \frac{\partial f}{\partial u_x} n_1 + \frac{\partial f}{\partial u_y} n_2 \right] v ds$$

- Dirichlet b.c.
  - $u$ is predefined at the boundary $\partial \Omega \rightarrow v(\partial \Omega) = 0$

- Neumann b.c.
  - derivative in the direction of normal $\frac{\partial u}{\partial n} = 0$

Example

Consider $F(u) = \int_\Omega \frac{1}{2} |\nabla u|^2 = \int_\Omega \frac{1}{2} (u_x^2 + u_y^2)$
Boundary conditions

- Using “per partes” on \( u(x, y) \), \( n(x, y) = [n_1(x, y), n_2(x, y)] \)
- Normal vector at the boundary \( \partial \Omega \)

\[
\frac{\partial}{\partial \varepsilon} F(u + \varepsilon v) = \int \cdot dx \, dy + \int_{\partial \Omega} \left[ \frac{\partial f}{\partial u_x} n_1 + \frac{\partial f}{\partial u_y} n_2 \right] v \, ds
\]

- Dirichlet b.c.
  \( u \) is predefined at the boundary \( \partial \Omega \rightarrow v(\partial \Omega) = 0 \)

- Neumann b.c.
  Derivative in the direction of normal \( \frac{\partial u}{\partial n} = 0 \)

**Example**

Consider \( F(u) = \int_\Omega \frac{1}{2} |\nabla u|^2 = \int_\Omega \frac{1}{2} (u_x^2 + u_y^2) \)

\[
\frac{\partial f}{\partial u_x} = u_x, \quad \frac{\partial f}{\partial u_y} = u_y
\]
Boundary conditions

- using “per partes” on \( u(x, y), \mathbf{n}(x, y) \equiv [n_1(x, y), n_2(x, y)] \)
  
  normal vector at the boundary \( \partial \Omega \)

\[
\frac{\partial}{\partial \varepsilon} F(u + \varepsilon v) = \int (\cdot) dx dy + \int_{\partial \Omega} \left[ \frac{\partial f}{\partial u_x} n_1 + \frac{\partial f}{\partial u_y} n_2 \right] v \, ds
\]

- **Dirichlet b.c.**
  
  \( u \) is predefined at the boundary \( \partial \Omega \rightarrow v(\partial \Omega) = 0 \)

- **Neumann b.c.**
  
  derivative in the direction of normal \( \frac{\partial u}{\partial \mathbf{n}} = 0 \)

**Example**

Consider \( F(u) = \int_\Omega \frac{1}{2} |\nabla u|^2 = \int_\Omega \frac{1}{2} (u_x^2 + u_y^2) \)

\[
\frac{\partial f}{\partial u_x} = u_x, \quad \frac{\partial f}{\partial u_y} = u_y
\]

\[
\frac{\partial f}{\partial u_x} n_1 + \frac{\partial f}{\partial u_y} n_2 = u_x n_1 + u_y n_2 = \frac{\partial u}{\partial \mathbf{n}} = 0
\]

**Variational Methods**
E-L equation example

- Smoothing functional:

\[ F(u) = \int_{\Omega} |\nabla u|^2 \, dx, \quad f = u_x^2 + u_y^2 \]
E-L equation example

- **Smoothing functional:**
  \[ F(u) = \int_{\Omega} |\nabla u|^2 \, dx , \quad f = u_x^2 + u_y^2 \]

- **E-L equation:**
  \[ F'(u) = -\Delta u = -u_{xx} - u_{yy} \]
More examples

- Total variation of an image function $u(x,y)$:

$$F(u) = \int_{\Omega} |\nabla u| \, dx, \quad f = \sqrt{u_x^2 + u_y^2}$$
More examples

- Total variation of an image function $u(x,y)$:

$$F(u) = \int_{\Omega} |\nabla u| \, dx, \quad f = \sqrt{u_x^2 + u_y^2}$$

- E-L equation:

$$\frac{\partial f}{\partial u} - \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial u_x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial u_y} \right)$$
More examples

- Total variation of an image function $u(x,y)$:
  \[ F(u) = \int_{\Omega} |\nabla u| \, dx , \quad f = \sqrt{u_x^2 + u_y^2} \]

- E-L equation:
  \[ \frac{\partial f}{\partial u} - \frac{\partial}{\partial x} \frac{\partial f}{\partial u_x} - \frac{\partial}{\partial y} \frac{\partial f}{\partial u_y} \]
  \[ - \frac{\partial}{\partial x} \frac{u_x}{\sqrt{u_x^2 + u_y^2}} - \frac{\partial}{\partial y} \frac{u_y}{\sqrt{u_x^2 + u_y^2}} = - \text{div} \left( \frac{\nabla u}{|\nabla u|} \right) \]
More examples

- Total variation of an image function $u(x,y)$:

$$F(u) = \int_{\Omega} \|\nabla u\| \, dx , \quad f = \sqrt{u_x^2 + u_y^2}$$

- E-L equation:

$$\frac{\partial f}{\partial u} - \frac{\partial}{\partial x} \frac{\partial f}{\partial u_x} - \frac{\partial}{\partial y} \frac{\partial f}{\partial u_y}$$

$$- \frac{\partial}{\partial x} \frac{u_x}{\sqrt{u_x^2 + u_y^2}} - \frac{\partial}{\partial y} \frac{u_y}{\sqrt{u_x^2 + u_y^2}} = - \text{div} \left( \frac{\nabla u}{|\nabla u|} \right)$$
More examples

- Total variation of an image function $u(x,y)$:
  \[ F(u) = \int_{\Omega} |\nabla u| \, dx, \quad f = \sqrt{u_x^2 + u_y^2} \]

- E-L equation:
  \[ \frac{\partial f}{\partial u} - \frac{\partial}{\partial x} \frac{\partial f}{\partial u_x} - \frac{\partial}{\partial y} \frac{\partial f}{\partial u_y} \]
  \[ - \frac{\partial}{\partial x} \frac{u_x}{\sqrt{u_x^2 + u_y^2}} - \frac{\partial}{\partial y} \frac{u_y}{\sqrt{u_x^2 + u_y^2}} = - \text{div} \left( \frac{\nabla u}{|\nabla u|} \right) \]
1 Introduction
   • Motivation
   • Derivation of Euler-Lagrange Equation
   • Variational Problem and P.D.E.
Steepest Descent

- Classical optimization problem

\[ g : \mathbb{R} \rightarrow \mathbb{R}, \quad \tilde{x} = \min_x g(x) \]
Steepest Descent

Classical optimization problem

\[ g : R \rightarrow R, \; \tilde{x} = \min_x g(x) \]

Must satisfy \( g'(\tilde{x}) = 0 \)
Steepest Descent

- Classical optimization problem

\[ g : \mathbb{R} \rightarrow \mathbb{R}, \quad \tilde{x} = \min_x g(x) \]

- Must satisfy \( g'(\tilde{x}) = 0 \)
- Imagine, analytical solution is impossible.
Steepest Descent

Classical optimization problem

\[ g : \mathbb{R} \rightarrow \mathbb{R}, \tilde{x} = \min_x g(x) \]

- Must satisfy \( g'(\tilde{x}) = 0 \)
- Imagine, analytical solution is impossible.
- Let us walk in the direction opposite to the gradient

\[ x_{k+1} = x_k - \alpha g'(x_k), \]

where \( \alpha \) is the step length
Steepest Descent

- Classical optimization problem

\[ g : R \rightarrow R, \quad \tilde{x} = \min_x g(x) \]

- Must satisfy \( g'(\tilde{x}) = 0 \)
- Imagine, analytical solution is impossible.
- Let us walk in the direction opposite to the gradient

\[ x_{k+1} = x_k - \alpha g'(x_k), \]

where \( \alpha \) is the step length
Steepest Descent

- Classical optimization problem
  \[ g : R \to R, \, \tilde{x} = \min_x g(x) \]

- Must satisfy \( g'(\tilde{x}) = 0 \)
- Imagine, analytical solution is impossible.
- Let us walk in the direction opposite to the gradient
  \[ x_{k+1} = x_k - \alpha g'(x_k), \]

where \( \alpha \) is the step length.
Steepest Descent

Classical optimization problem

\[ g : \mathbb{R} \rightarrow \mathbb{R}, \hat{x} = \min_{x} g(x) \]

Must satisfy \( g'(\hat{x}) = 0 \)
Imagine, analytical solution is impossible.
Let us walk in the direction opposite to the gradient

\[ x_{k+1} = x_k - \alpha g'(x_k), \]

where \( \alpha \) is the step length
Steepest Descent

- Classical optimization problem

\[ g : \mathbb{R} \rightarrow \mathbb{R}, \tilde{x} = \min_{x} g(x) \]

- Must satisfy \( g'(\tilde{x}) = 0 \)

- Imagine, analytical solution is impossible.

- Let us walk in the direction opposite to the gradient

\[ x_{k+1} = x_k - \alpha g'(x_k), \]

where \( \alpha \) is the step length
Steepest Descent

\[ \forall \alpha \]

\[ \frac{x_{k+1} - x_k}{\alpha} = -g'(x_k), \]

Finding the solution with the steepest-descent method is equivalent to solving PDE:
Steepest Descent

\[ \forall \alpha \]

\[ \frac{x_{k+1} - x_k}{\alpha} = -g'(x_k), \]

Define \( x(t) \) as a function of time such that \( x(t_k) = x_k \) and
\[ t_{k+1} = t_k + \alpha \]

\[ \frac{dx}{dt}(t_k) = \lim_{\alpha \to 0} \frac{x(t_k + \alpha) - x(t_k)}{\alpha} = \lim_{\alpha \to 0} \frac{x_{k+1} - x_k}{\alpha} = -g'(x_k) \]
Steepest Descent

∀ \alpha

\frac{x_{k+1} - x_k}{\alpha} = -g'(x_k),

Define \( x(t) \) as a function of time such that \( x(t_k) = x_k \) and \( t_{k+1} = t_k + \alpha \)

\frac{dx}{dt}(t_k) = \lim_{\alpha \to 0} \frac{x(t_k + \alpha) - x(t_k)}{\alpha} = \lim_{\alpha \to 0} \frac{x_{k+1} - x_k}{\alpha} = -g'(x_k)

Finding the solution with the steepest-descent method is equivalent to solving P.D.E.:

\frac{dx}{dt} = -g'(x)
Variational problem

\[ \tilde{u} = \min_u F(u(x)) \]
Variational problem

\[ \tilde{u} = \min_u F(u(x)) \]

Must satisfy E-L equation

\[ \Rightarrow F'(\tilde{u}) = 0 \]
Variational problem

$$\tilde{u} = \min_u F(u(x))$$

Must satisfy E-L equation

$$\Rightarrow F'(\tilde{u}) = 0$$

Find the solution with the steepest-descent method

$$u_{k+1} = u_k - \alpha F'(u_k),$$

where $\alpha$ is the step length and must be determined
Variational problem

\[ \hat{u} = \min_u F(u(x)) \]

Must satisfy E-L equation

\[ \Rightarrow F'(\hat{u}) = 0 \]

Find the solution with the steepest-descent method

\[ u_{k+1} = u_k - \alpha F'(u_k), \]

where \( \alpha \) is the step length and must be determined

\[ \forall \alpha \]

\[ \frac{u_{k+1} - u_k}{\alpha} = -F'(u_k), \]
Make \( u \) also function of time, i.e., \( u(x, t) \)

\[ u_k(x) \equiv u(x, t_k) \]

and \( t_{k+1} = t_k + \alpha \)

\[ \lim_{\alpha \to 0} \frac{u_{k+1} - u_k}{\alpha} \equiv \frac{\partial u}{\partial t}(x, t_k) \]
Make \( u \) also function of time, i.e., \( u(x, t) \)

\[
u_k(x) \equiv u(x, t_k)
\]

and \( t_{k+1} = t_k + \alpha \)

\[
\lim_{\alpha \to 0} \frac{u_{k+1} - u_k}{\alpha} \equiv \frac{\partial u}{\partial t}(x, t_k)
\]

Solving the variational problem with the steepest-descent method is equivalent to solving P.D.E.: 

\[
\frac{\partial u}{\partial t} = -F'(u)
\]

+ boundary conditions.
Steepest descent

$g(x)$

$x_0$
Steepest descent

$g(x)$

$x_0$
Steepest descent

\[ g(x) \]

\[ x_0 \]
Steepest descent

\[ g(x) \]

\[ x_0 \]

\[ a \]

\[ b \]

\[ u(x) \]
Steepest descent

\[ g(x) \]

\[ x_0 \]

\[ a \]

\[ b \]

\[ u(x) \]
Steepest descent

\[ g(x) \]

\[ x_0 \]

\[ u(x) \]

\[ a \]

\[ b \]
Steepest descent

$g(x)$

$x_0$

$u(x)$

$a$

$b$
### Differential Calculus vs Variational Calculus

<table>
<thead>
<tr>
<th>Problem Spec.</th>
<th>Differential Calculus</th>
<th>Variational Calculus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem Spec.</td>
<td>function</td>
<td>function of function</td>
</tr>
<tr>
<td>Necess. Cond.</td>
<td>1st derivative = 0</td>
<td>1st variation = 0</td>
</tr>
<tr>
<td>Result</td>
<td>one number (or vector)</td>
<td>function</td>
</tr>
</tbody>
</table>
Solving PDE’s is equivalent to optimization of integral functionals
Solving PDE’s is equivalent to optimization of integral functionals

\[ u_t - F'(u) = 0 \iff \min F(u) \]
Solving PDE’s is equivalent to optimization of integral functionals

\[ u_t - F'(u) = 0 \iff \min F(u) \]

Example

\[ u_t = \Delta u \iff \min \int_{\Omega} |\nabla u|^2 \]
Solving PDE’s is equivalent to optimization of integral functionals

\[ u_t - F'(u) = 0 \iff \min F(u) \]

Example

\[ u_t = \Delta u \iff \min \int_{\Omega} |\nabla u|^2 \]

Does every PDE have its corresponding optimization problem?